

SUPER MORITA THEORY

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ABSTRACT. We develop the basics of Morita theory for super rings. As an application, we produce a more explicit super Morita equivalence in the case of super Azumaya algebras.

1. INTRODUCTION

Two associative rings R and S with unit are said to be *Morita equivalent* if the categories \mathfrak{M}_R and \mathfrak{M}_S of right R - (resp. S -) modules are equivalent.

The simplest and most prototypical example of Morita equivalence is the equivalence between the category of R -modules and the category of $\mathbb{M}_n(R)$ -modules, where $\mathbb{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in some associative ring R . (This example is treated in detail in Ch. 17 of [8]).

Many interesting properties of rings (such as centers, K-theory, G-theory etc.) remain invariant under Morita equivalence, making the theory of Morita equivalences a central tool in many areas of algebra.

The purpose of this work is to extend the basic theory of Morita equivalences to super rings, i.e. rings R with a \mathbb{Z}_2 -grading such that the multiplication is compatible with the grading. The category of R -supermodules is that whose objects are \mathbb{Z}_2 -graded R -modules, and whose morphisms are parity-preserving R -module homomorphisms.

We now briefly outline the contents of this paper. In the second section, we explain the supermodule theory needed for the remainder of the paper, including the definitions of projective module and generator.

The third section is the main portion of this paper and it contains the proofs of the basic Morita theorems for supermodules. Our treatment is modeled on the discussion of ungraded Morita theory in Ch. 18 of [8]; one may see Ch. 1 of [1] for a more abstract formulation and development of this theory.

In the final section, we apply the super Morita theory developed in the third section to the context of *super Azumaya algebras*, and show that for a super Azumaya algebra A , the Morita equivalence may be

realized more explicitly, in terms of the *supercommutant* of an (A, A) -bimodule M .

In the ungraded case, the corresponding results on ordinary Azumaya algebras may be found in III.5 of [6], where they are derived as a corollary of more general considerations. However, we have preferred to give a more self-contained and concrete treatment of this material, better suited to our own purposes.

In another paper [7], we develop the theory of Π -invertible sheaves in supergeometry. Our main motivation for the present work was ultimately to explain the existence of a “product” structure on Π -invertible sheaves in terms of the “super skew field”: the super Azumaya algebra $\mathbb{D} = k[\theta]$, where θ is odd, $\theta^2 = -1$, for k an algebraically closed field of characteristic $\neq 2$.

We note that similar considerations are briefly discussed in the paper [5], in the context of quiver Hecke superalgebras.

2. MODULE THEORY FOR SUPER RINGS

In this section R denotes a (not necessarily commutative) associative super ring with unit, that is, a \mathbb{Z}_2 -graded ring $R = R_0 \oplus R_1$ whose multiplication is compatible with the grading:

$$R_i \cdot R_j \subseteq R_{i+j}$$

where $i, j \in \mathbb{Z}_2$.

A *supermodule* for a super ring R is a \mathbb{Z}_2 -graded R -module $M = M_0 \oplus M_1$ such that the R -module structure is compatible with the grading on M :

$$R_i \cdot M_j \subseteq M_{i+j}$$

The *parity* of a homogeneous element of a super ring R , or R -supermodule M , is defined as:

$$\begin{aligned} |x| &= 0 \text{ if } x \text{ is even} \\ &1 \text{ if } x \text{ is odd} \end{aligned}$$

R -supermodule homomorphisms are simply R -module homomorphisms which preserve the grading (i.e., the parity). The set of R -supermodule homomorphisms between supermodules M and N is denoted $\text{Hom}_R(M, N)$ (for simplicity of notation, we will often drop the subscript R and write

$Hom(M, N)$ when the ring R under consideration is clear). We will often refer to $Hom(M, N)$ as the *categorical Hom*.

We will also have occasion to consider the collection of all R -module homomorphisms between M and N (regarding both as *ungraded* modules), which we denote by $\underline{Hom}_R(M, N)$. We refer to $\underline{Hom}_R(-, -)$ as the *internal Hom*. As before, we will often drop the subscript R when the ring under consideration is clear.

$\underline{Hom}(M, N)$ is endowed with a natural \mathbb{Z}_2 -grading: the even part $(\underline{Hom}(M, N))_0$ consists of the parity-preserving homomorphisms, and the odd $(\underline{Hom}(M, N))_1$ consists of the parity-reversing homomorphisms. (The even homomorphisms are precisely the R -supermodule homomorphisms in the sense just defined).

We may make the collection of right (resp. left) R -modules into a category \mathfrak{M}_R (resp. ${}_R\mathfrak{M}$) by taking the morphisms between two R -modules M and N to be the R -supermodule homomorphisms $Hom(M, N)$. This is the reason we refer to $Hom(-, -)$ as the categorical *Hom*.

The word “homomorphism” may denote either elements of Hom or of \underline{Hom} , but we will reserve the term *morphism* solely for elements of Hom . Likewise, the terms *monomorphism*, *epimorphism*, *isomorphism*, etc. will be understood to refer only to elements of Hom unless otherwise specified.

From now on we will drop the prefix “super” and refer to objects of \mathfrak{M}_R and ${}_R\mathfrak{M}$ simply as *modules*.

Some conventions: homomorphisms $f : M \rightarrow N$ of left R -modules will occasionally be written to act on the right of M : thus we will sometimes write xf for $f(x)$. In this case we will use \bullet to denote composition of homomorphisms: $x(g \bullet f)$ for $f(g(x))$.

Note that with this convention, we have:

$$(rx)f = (-1)^{|r||f|}r(xf)$$

Given super rings S, R , we may define the category of (S, R) -super bimodules ${}_S\mathfrak{M}_R$ in the obvious way (the corresponding definitions of categorical *Hom* and internal *Hom* should by now be clear to the reader).

We define the dual of a right (resp. left) R -module M , denoted M^* , to be the left (resp. right) R -module $\underline{Hom}(M, R)$, with the R -module structure given by left (resp. right) multiplication in R :

$$\begin{aligned}(rf)(x) &:= r[f(x)] \\ x(fr) &:= (-1)^{|x||r|}(xf)r\end{aligned}$$

Finally, if M is a right R -module and N is a left R -module, the tensor product $M \otimes_R N$ is defined to be the usual tensor product of M and N , but endowed with the grading

$$(M \otimes_R N)_k = \bigoplus_{i+j=k} M_i \otimes_R N_j$$

If $f : M \rightarrow M', g : N \rightarrow N'$ are right (resp left) R -module homomorphisms, then the induced map $f \otimes g : M \otimes_R N \rightarrow M' \otimes_R N'$ is:

$$(f \otimes g)(m \otimes n) := (-1)^{|g||m|} fm \otimes ng$$

We define the category of *superabelian groups* to be the category of \mathbb{Z} -supermodules, with \mathbb{Z} considered as a (purely even) supercommutative ring: that is, the objects of the category are \mathbb{Z}_2 -graded abelian groups, with the morphisms being the parity-preserving group homomorphisms. (There should be no confusion with the completely different term “supergroup”). The internal *Hom* of the category of superabelian groups consists of all group homomorphisms, and we denote the category of superabelian groups by (SAb) .

Using this terminology, we note that $M \otimes_R N$ has *a priori* only the structure of a superabelian group.

If we suppose in addition that M, M' are (S, R) -bimodules, N, N' are (R, T) -bimodules for S, R super rings, and f, g are (S, R) (resp. (R, T) -bimodule homomorphisms, then one may verify that $f \otimes g : M \otimes_R N \rightarrow M' \otimes_R N'$ defined as above is an (S, T) -bimodule homomorphism.

Next we discuss the *opposite* of a super ring R . This is the super ring R^o whose underlying set is equal to that of R , and whose multiplication is given by:

$$x \cdot_o y := (-1)^{|x||y|} y \cdot x$$

for $x, y \in R$, where the multiplication on the right hand side is that of R .

A left (resp. right) R -module M may be canonically converted into a right (resp. left) R^o -module by defining the right R - (resp. left R^o)-action to be:

$$m \cdot_o r := (-1)^{|r||m|} r \cdot m$$

(resp.

$$r \cdot_o m := (-1)^{|r||m|} m \cdot r).$$

One checks readily that this recipe sends left (resp. right) R -module homomorphisms to right (resp. left) R^o -module homomorphisms, and that it preserves the parity of homomorphisms. In particular, it defines a natural functor $\cdot^o : {}_R\mathfrak{M} \rightarrow \mathfrak{M}_{R^o}$ (resp. $\mathfrak{M}_R \rightarrow {}_{R^o}\mathfrak{M}$). The reader may verify that $(\cdot^o)^2$ is the identity functor (the key point is that $(R^o)^o = R$ as super rings), hence \cdot^o is an equivalence of categories. We may therefore convert any statement about right (resp. left) R -modules into a corresponding statement about left (resp. right) R^o -modules in a completely natural way. In particular, $\underline{End}_R(P) = \underline{End}_{R^o}(P^o)$ as super rings.

One readily extends \cdot^o to categories of bimodules as well, and checks that $\cdot^o : {}_S\mathfrak{M}_R \rightarrow {}_{R^o}\mathfrak{M}_{S^o}$ is an equivalence of categories as before.

Suppose $M \in \mathfrak{M}_R$ and $N \in {}_R\mathfrak{M}$. Using the universal property of the super tensor product (cf. [4]), the reader may verify that there is a natural isomorphism of superabelian groups $(M \otimes_R N)^o \rightarrow N^o \otimes_{R^o} M^o$ given by $m \otimes_R n \mapsto (-1)^{|m||n|} n \otimes_{R^o} m$. If M is an (S, R) -bimodule and N is an (R, T) -bimodule, then this isomorphism is an isomorphism in the category of (T^o, S^o) -bimodules.

2.1. Projective modules. The definition of projective module is exactly the same as that in the classical case:

Definition 2.1.1. A (right) R -module P over a super ring R is said to be *projective* if $Hom_R(P, -) : \mathfrak{M}_R \rightarrow (Ab)$ is an exact functor; that is, if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of R -modules, then the induced sequence of abelian groups:

$$0 \rightarrow Hom(P, A) \rightarrow Hom_R(P, B) \rightarrow Hom_R(P, C) \rightarrow 0$$

is short exact.

Remark. 1) $\text{Hom}(M, -)$ is a left exact functor for any module M , that is, for an exact sequence:

$$0 \rightarrow A \rightarrow B \rightarrow C$$

the sequence

$$0 \rightarrow \text{Hom}(P, A) \rightarrow \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C)$$

is exact. The key property of a projective module P is that if $B \rightarrow C$ is an epimorphism, $\text{Hom}(P, B) \rightarrow \text{Hom}(P, C)$ is also an epimorphism.

2) Note that the definition of projective module involves the categorical Hom and not internal $\underline{\text{Hom}}$; as we shall see, this situation is the precise converse to the definition of generator, which involves internal $\underline{\text{Hom}}$ and not categorical Hom .

An R -module M is *free* if it has a homogeneous R -basis $\{e_i | f_j\}$ $i \in I$, $j \in J$, with e_i even, f_j odd. If $|I| = m$ and $|J| = n$ are both finite, then we define the *rank* of M to be the pair of integers $m|n$. It is left to the reader to check that a free R -module satisfies the usual sort of universal property. Having made this definition, the proof of the following proposition is also identical to that in the purely even case:

Proposition 2.1.2. *A right R -module P is projective if and only if there exist a free module F and morphisms $\pi : F \rightarrow P$, $i : P \rightarrow F$ such that $\pi \circ i = \text{id}_P$.*

We say that i is a *splitting* and that i *splits* π . This proposition is equivalent to the assertion that P is projective iff P is isomorphic to a direct summand of some free module F , and shows that our definition of projective module is entirely equivalent to that given in Appendix B of [2].

We have the following characterization of projective R -modules, the super Dual Basis Lemma.

Proposition 2.1.3. *Let R be an associative super ring. A right R -module P is projective if and only if there exist a family of homogeneous elements $\{a_i, b_j : i \in I, j \in J\} \subseteq P$ (a_i even, b_j odd) and homogeneous linear functionals $\{f_i, g_j : i \in I, j \in J\} \subseteq P^*$ (f_i even, g_j odd) such*

that, for any homogeneous $c \in P$, $f_i(c) = 0, g_j(c) = 0$ for almost all i, j , and $c = \sum_i a_i f_i(c) + \sum_j b_j g_j(c)$.

Proof. Suppose such a_i 's, b_j 's and f_i 's, g_j 's exist. Consider the epimorphism h from the free module $F := \bigoplus d_i R \oplus \bigoplus e_j R$ to P defined by $h(d_i) = a_i, h(e_j) = b_j$. (Here the d_i are even, the e_j odd). Then the map $k : P \rightarrow F$ given by $h(c) = \sum_i d_i f_i(c) + \sum_j e_j g_j(c)$ is a morphism splitting h , hence P is projective.

Conversely, suppose P is projective, and fix an epimorphism h from a free module $\bigoplus d_i R \oplus \bigoplus e_j R$ onto P . Since P is projective, there exists a splitting morphism $k : P \rightarrow F$, and for homogeneous c , we may write $k(c) = \sum_i d_i f_i(c) + \sum_j e_j g_j(c)$.

One checks that f_i, g_j are R -linear and homogeneous, and that f_i and g_i are zero for all but finitely many i, j .

Applying h to both sides of the previous equation, we have $c = \sum_i a_i f_i(c) + \sum_j b_j g_j(c)$, where $a_i := h(d_i), b_j := h(e_j)$. □

Remark. The definition of a projective left R -module is entirely analogous to that for right modules, and all the results proven in this section are also true in the category of left R -modules; the proofs may be easily supplied by the reader.

2.2. Generators. Suppose $I \subseteq R$ is a homogeneous two-sided ideal in a super ring R . Then the quotient ring R/I is naturally a right (left) R -module with grading induced from R , and the quotient map $R \rightarrow R/I$ is an R -module epimorphism.

Definition 2.2.1. Let P be a right R -module. The *trace ideal* of P , denoted $tr(P)$, is the module:

$$tr(P) := \sum g(P)$$

where the sum is taken over all homogeneous $g \in P^* = \underline{Hom}(P, R)$.

(We could just as well have left out the adjective “homogeneous” without affecting this definition at all, but what we have done makes the proof of Prop. 2.3.2 a little shorter.) One checks readily that $tr(P)$ is a homogeneous two-sided ideal of R . We now make the following important definition:

Definition 2.2.2. Let P be a right R -module over an associative super ring R . We say P is a *generator* in \mathfrak{M}_R if $\underline{Hom}_R(P, -) : \mathfrak{M}_R \rightarrow (SAb)$ is a faithful functor, i.e. for each R -morphism $f : M \rightarrow N$, $f_* = 0 \implies f = 0$, where $f_* : \underline{Hom}_R(P, M) \rightarrow \underline{Hom}_R(P, N)$ denotes the morphism in (SAb) induced by composition with f .

Remark. Here the super Morita theory diverges in a crucial way from the ungraded theory: whereas in ordinary module theory, categorical Hom and internal \underline{Hom} are the same thing, here they are different, and one must take careful note that it is the categorical Hom that appears in the definition of projective module, while it is the internal \underline{Hom} that appears in the definition of generator.

2.3. Parity reversal. We recall some facts about the parity reversal functor $\Pi : \mathfrak{M}_R \rightarrow \mathfrak{M}_R$ (resp. $\Pi : {}_R \mathfrak{M} \rightarrow {}_R \mathfrak{M}$). Given a (left or right) R -module M , the underlying set of ΠM is equal to the underlying set of M , but endowed with the reverse grading:

$$(\Pi M)_i = M_{i+1}$$

for $i \in \mathbb{Z}_2$. ΠM is endowed with natural right (resp. left) R -module structures. The right R -module structure on ΠM is defined to be the same as that on M :

$$m \cdot_{\Pi} r := m \cdot r$$

The left R -module structure is defined by:

$$r \cdot_{\Pi} m := (-1)^{|r|} r \cdot m$$

for homogeneous elements $r \in R, m \in \Pi M$. One checks that these indeed define R -module structures on ΠM .

Although Π is of course the identity when viewed purely as a map of sets, given an element m in the set M , we will occasionally write $\Pi(m)$ for emphasis when we regard m as an element of the module ΠM . With this convention, $\Pi(\Pi(m)) = m$, hence we may write formally $\Pi^2 = id$ (we will justify this when we prove that $\Pi(\Pi M) = M$ as R -modules).

With this convention, the above definitions of the R -module structures may be rewritten as:

$$\begin{aligned}\Pi(r \cdot m) &= (-1)^{|r|} r \cdot_{\Pi} \Pi(m) \\ \Pi(m \cdot r) &= \Pi(m) \cdot_{\Pi} r\end{aligned}$$

Given $f : M \rightarrow N$ a homogeneous R -homomorphism, we may define associated homomorphisms $\Pi f : M \rightarrow \Pi N$ and $f\Pi : \Pi M \rightarrow N$

$$\begin{aligned}(\Pi f)(m) &:= \Pi(f(m)) \\ (f\Pi)(\Pi m) &:= f(m)\end{aligned}$$

The reader will check that these are indeed R -module morphisms. In the next proposition, we collect some basic facts, which can be mostly found in Ch.3, 1.5 of [9], relating Π to module homomorphisms.

Proposition 2.3.1. *Let M and N be right (left) R -modules. Then*

- $\Pi(\Pi M) = M$ as right (left) R -modules.
- *There are odd isomorphisms of superabelian groups:*

$$\begin{aligned}\Pi \cdot : \underline{Hom}_R(M, N) &\rightarrow \underline{Hom}_R(M, \Pi N) \\ f &\mapsto \Pi f \\ \cdot \Pi : \underline{Hom}_R(M, N) &\rightarrow \underline{Hom}_R(\Pi M, N) \\ f &\mapsto f\Pi\end{aligned}$$

Proof. To prove that $\Pi(\Pi M) = M$, one first notes that the gradings on M and $\Pi(\Pi M)$ are the same:

$$\begin{aligned}[\Pi(\Pi M)]_i &= (\Pi M)_{i+1} \\ &= M_i\end{aligned}$$

Now we check they have the same R -module structures. The case of right modules is trivial, so we only need check the case of left modules.

$$\begin{aligned}r \cdot_{\Pi\Pi} m &= (-1)^{|r|} r \cdot_{\Pi} m \\ &= r \cdot m\end{aligned}$$

Thus $\Pi(\Pi M) = M$ in ${}_R\mathfrak{M}$ and \mathfrak{M}_R , as desired.

We now discuss the second item. Let $f \in \underline{Hom}(M, N)$ be homogeneous. Then $\Pi f \in \underline{Hom}(M, \Pi N)$ is homogeneous of the opposite parity. Hence $f \rightarrow \Pi f$ is an odd map. That it is a homomorphism of abelian groups is clear. Similarly, if $g : M \rightarrow \Pi N$, then $g\Pi : M \rightarrow \Pi(\Pi N) = N$ is a homomorphism of the opposite parity, hence $g \mapsto \Pi g$ gives an odd homomorphism of abelian groups $\underline{Hom}(M, \Pi N) \rightarrow \underline{Hom}(M, N)$ and it is left to the reader to check that it is inverse to $f \mapsto \Pi f$.

The proof for $\cdot \Pi$ proceeds similarly and is also left to the reader. \square

We now give several equivalent characterizations of a generator:

Proposition 2.3.2. *Let P be a right R -module over an associative super ring R . The following are equivalent:*

- (1) P is a generator in \mathfrak{M}_R .
- (2) $tr(P) = R$.
- (3) R is a direct summand of a finite direct sum $\oplus_i P \oplus_j \Pi P$.
- (4) R is a direct summand of a direct sum $\oplus_i P \oplus_j \Pi P$.
- (5) Every $M \in \mathfrak{M}_R$ is an epimorphic image of some direct sum $\oplus_i P \oplus_j \Pi P$.

Proof. 1) \implies 2). Suppose $I := tr(P) \neq R$. Then the quotient map $R \rightarrow R/I$ is nonzero in \mathfrak{M}_R , hence by the hypothesis that P is a generator, there is some $g \in \underline{Hom}_R(P, R)$ such that $P \xrightarrow{g} R \rightarrow R/I$ is nonzero. But then $g(P) \not\subseteq I$, contradicting the definition of g .

2) \implies 3). By 2), there exist $f_1, \dots, f_m, g_1, \dots, g_n$, with f_i even and g_j odd, such that $\sum_i f_i(p_i) + \sum_j g_j(q_j) = 1$. By taking homogeneous components of this equation, we may assume that the p_i are even, q_j are odd. Then $(f_1, \dots, f_m, g_1\Pi, \dots, g_n\Pi) : P \oplus \dots \oplus P \oplus \Pi P \oplus \dots \oplus \Pi P \rightarrow R$ is a split epimorphism, with splitting given by $1 \mapsto (p_1, \dots, p_m, q_1, \dots, q_n)$, whence 3).

3) \implies 4). Tautological.

4) \implies 5). Follows easily, since M is an epimorphic image of a free module.

5) \implies 1). Suppose $f : M \rightarrow N$ is a nonzero morphism. By 5) there exists an epimorphism $\oplus_i P \oplus_j \Pi P \rightarrow M$. The composition $\oplus_i P \oplus_j \Pi P \rightarrow M \xrightarrow{f} N$ is clearly nonzero. Hence, either for some i , $P_i = P \xrightarrow{g} M \xrightarrow{f} N$ is nonzero, or for some j , $\Pi P_j = \Pi P \xrightarrow{h} M \xrightarrow{f} N$ is nonzero. But by Prop. 2.3.1, $h : \Pi P \rightarrow M$ may be regarded as a

homomorphism $h\Pi : \Pi(\Pi P) = P \rightarrow M$ of the opposite parity, and the composition $P \xrightarrow{h\Pi} M \xrightarrow{f} N$ is nonzero. In either case, we have proven that $\underline{Hom}_R(P, -)$ is faithful, as desired. \square

Remark. Just as Lam points out in [8] for the classical case, the notions of finitely generated projective module and generator for a super ring are complementary: P is finitely generated projective iff P is a direct summand of $R^{m|n}$ for some m, n ; P is a generator iff R (regarded naturally as a free R module of rank $1|0$ with basis $\{1\}$) is a direct summand of $P^{m|n} = P^m \oplus (\Pi P)^n$ for some m, n . This suggests that combining the two conditions will yield an interesting notion:

Definition 2.3.3. An R -module P is a *progenerator* iff it is a finitely generated projective generator.

Just as in the ungraded case, it is the concept of progenerator which is crucial to the theory of Morita equivalences.

We will need to show that being a progenerator is a categorical property. More precisely:

Proposition 2.3.4. *Let R, S be super rings, and $F : \mathfrak{M}_R \rightarrow \mathfrak{M}_S$ an equivalence of categories. If P is a progenerator in \mathfrak{M}_R , then $F(P)$ is a progenerator in \mathfrak{M}_S .*

Proof. Suppose P is a progenerator in \mathfrak{M} . The statement that P is projective is equivalent to:

The functor $Hom_R(P, -)$ is exact.

Since F is an equivalence of categories, it takes short exact sequences in \mathfrak{M}_R to short exact sequences in \mathfrak{M}_S , hence $F(P)$ is projective if P is projective.

One checks that the statement that P is finitely generated is equivalent to:

For any family of submodules $\{N_i : i \in I\}$ of P , if $\sum_{i \in I} N_i = M$, then $\sum_{i \in J} N_i = M$ for some finite subset $J \subseteq I$.

This is clearly preserved by a category equivalence, so $F(P)$ is finitely generated if P is finitely generated. Finally, the statement that P is a generator is:

The functor $\underline{Hom}_R(P, -)$ is faithful.

This is again preserved by category equivalences, so $F(P)$ is a generator if P is a generator. \square

Remark. As in the previous section, all definitions and theorems that were stated for right R -modules also hold for categories ${}_R\mathfrak{M}$ of left R -modules. Proofs are left to the reader.

3. SUPER MORITA THEORY

3.1. The super Morita context. Let R be a super ring, P a right R -module, and $Q := P^* = \underline{Hom}_R(P, R)$, $S := \underline{End}_R(P)$, with Q, S both acting on the left on P . Thus P becomes an (S, R) -bimodule.

As in the classical case, we define the left action of R on Q by $(rq)(p) := r(qp)$ (“ RQP -associativity”) and the right action of S by $(qs)(p) := q(sp)$ (“ QSP -associativity”), thus making Q an (R, S) bimodule in a natural way.

There are also several important pairings involving Q and P .

Lemma 3.1.1. *Let $p, p' \in P, q, q' \in Q$. Define the pairings:*

$$\begin{aligned} Q \times P &\rightarrow R \\ (q, p) &\mapsto qp := q(p) \end{aligned}$$

$$\begin{aligned} P \times Q &\rightarrow S \\ (p, q) &\mapsto pq \end{aligned}$$

where $(pq)(p') := p(qp')$. Then:

- 1) $(q, p) \mapsto qp$ defines an (R, R) -morphism $\alpha : Q \otimes_S P \rightarrow R$;
- 2) $(p, q) \mapsto pq$ defines an (S, S) -morphism $\beta : P \otimes_R Q \rightarrow S$.

Proof. As P is an (S, R) -bimodule and Q an (R, S) -bimodule, the tensor product $Q \otimes_S P$ makes sense and is an (R, R) -bimodule. QSP -associativity implies the S -bilinearity $(qs, p) = (q, sp)$ of the first pairing, hence it induces a morphism (of superabelian groups) $\alpha : Q \otimes_S P \rightarrow R$.

Now α is given by $q \otimes p \mapsto qp$. Since we have $|q \otimes p| = |q| + |p| = |qp|$, we see $\alpha : Q \otimes_S P \rightarrow R$ preserves parity. That α is actually an (R, R) -morphism is a consequence of RQP -associativity and

QPR -associativity, respectively. We have already discussed RQP -associativity $r(qp) = (rq)p$, and QPR -associativity $(qp)r = q(pr)$ is a restatement of the R -linearity of $q \in P^*$.

Similarly, the proof of 2) is an easy consequence of PRQ -, SPQ -, and PQS -associativities, and is left to the reader. The only thing to note is that $p \otimes q \mapsto pq$ is a parity preserving map (the argument is the same as that given for α), hence β is a morphism of superabelian groups. \square

Definition 3.1.2. The *super Morita context* associated to P_R is the 6-tuple

$$(R, P, Q, S; \alpha, \beta).$$

Remark. In order to carry out this discussion for left R -modules, we have to make the following conventions. If $P \in {}_R\mathfrak{M}$, then $S = \underline{\text{End}}({}_R P)$ still acts on the left. We convert the left S -action into a right S^o -action in the canonical way:

$$ps := (-1)^{|s||p|}s(p)$$

so that P becomes an (R, S^o) -bimodule. Homomorphisms in $Q = \underline{\text{Hom}}(P, R)$ now act on P on the right, as discussed in section 2. Then Q becomes an (S^o, R) -bimodule, with the right action of R on Q given by right multiplication in R , and the left action of S^o given by:

$$p(sq) := (ps)q$$

In what follows, we could reformulate and reprove everything that we do for categories of right modules for categories of left modules. This is a lengthy exercise, but it requires nothing new except a change of notation, and in the author's view, little would be gained in so doing. (The reader is invited to provide these proofs to his own satisfaction). Instead, we use the category equivalence \cdot^o to turn statements about left R -modules into statements about right R^o -modules, therefore reducing the results for left modules to the already-proven case of right modules.

We shall now proceed to prove some important facts about the super Morita context.

Proposition 3.1.3. *Let $P \in \mathfrak{M}_R$, $(R, P, Q, S; \alpha, \beta)$ the super Morita context associated to P . Then:*

- (1) *P is a generator iff α is onto.*

(2) Suppose P is a generator. Then

- a) $\alpha : Q \otimes_S P \rightarrow R$ is an (R, R) isomorphism.
- b) $Q \cong \underline{Hom}_S(P, S)$ as (R, S) -bimodules.
- c) $P \cong \underline{Hom}_S(Q, S)$ as (S, R) -bimodules.
- d) $R \cong \underline{End}_S({}_S P)^o \cong \underline{End}_S(Q_S)$ as super rings.

Proof. 1) follows from Prop. 2.3.2. For 2), suppose P_R is a generator, then we have an equation $1 = \sum_i q_i p_i + \sum_j \tilde{q}_j \tilde{p}_j$ where $q_i \in Q, p_i \in P$ are even and $\tilde{q}_j \in Q, \tilde{p}_j \in P$ are odd. For 2a), suppose $\sum_k q'_k \otimes p'_k$ is homogeneous, and that $\alpha(\sum_k q'_k \otimes p'_k) = \sum_k q'_k p'_k = 0$. Then

$$\begin{aligned}
\sum_k q'_k \otimes p'_k &= \sum_{k,i,j} (q_i p_i + \tilde{q}_j \tilde{p}_j) q'_k \otimes p'_k \\
&= \sum_{k,i,j} [q_i (p_i q'_k) + \tilde{q}_j (\tilde{p}_j q'_k)] \otimes p'_k \\
&= \sum_{i,k} [q_i \otimes (p_i q'_k) p'_k] + \sum_{j,k} \tilde{q}_j \otimes (\tilde{p}_j q'_k) p'_k \\
&= \sum_i q_i \otimes p_i \left(\sum_k q'_k p'_k \right) + \sum_j \tilde{q}_j \otimes \tilde{p}_j \left(\sum_k q'_k p'_k \right) \\
&= 0
\end{aligned}$$

To prove 2b), we define a map $\lambda : Q \rightarrow \underline{Hom}_S({}_S P, {}_S S)$ by $p \cdot \lambda(q) := (-1)^{|p||q|} p q \in S$. That $\lambda(q)$ is parity-preserving is clear: $|p \cdot \lambda(q)| = |p q| = |p| + |q|$. That $\lambda(q) \in \underline{Hom}_S(P, S)$ is a consequence of SPQ -associativity $(sp)q = s(pq)$:

$$\begin{aligned}
(sp) \cdot \lambda(q) &= (-1)^{|sp||q|} (sp)q \\
&= (-1)^{(|s|+|p|)|q|} (sp)q \\
&= (-1)^{(|s|+|p|)|q|} s(pq) \\
&= (-1)^{|s||q|} s(p \cdot \lambda(q)) \\
&= (-1)^{|s||\lambda(q)|} s(p \cdot \lambda(q))
\end{aligned}$$

Hence λ is an even homomorphism. We now show that λ is injective: suppose $pq = 0$ for all $p \in P$. Then, since $1_R = \sum_i q_i p_i + \sum_j \tilde{q}_j \tilde{p}_j$, we have:

$$\begin{aligned}
q &= 1_R q \\
&= \left[\sum_i q_i p_i + \sum_j \tilde{q}_j \tilde{p}_j \right] q \\
&= \sum_i q_i (p_i q) + \sum_j \tilde{q}_j (\tilde{p}_j q) \\
&= \sum_i q_i (0) + \sum_j \tilde{q}_j (0) \\
&= 0
\end{aligned}$$

We now prove that λ is surjective: suppose $f \in \underline{Hom}_S(P, S)$. We have:

$$\begin{aligned}
pf &= \left[p \left(\sum_i q_i p_i + \sum_j \tilde{q}_j \tilde{p}_j \right) \right] f \\
&= \sum_i ((pq_i)p_i)f + \sum_j ((p\tilde{q}_j)\tilde{p}_j)f \\
&= \sum_i (-1)^{|pq_i||f|} (pq_i)(p_i f) + \sum_j (-1)^{|p\tilde{q}_j||f|} (p\tilde{q}_j)(\tilde{p}_j f) \\
&= p \left(\sum_i (-1)^{|pq_i||f|} q_i(p_i f) + \sum_j (-1)^{|p\tilde{q}_j||f|} \tilde{q}_j(\tilde{p}_j f) \right) \\
&= p \left(\sum_i (-1)^{|p||f|} q_i(p_i f) + \sum_j (-1)^{(|p|+1)|f|} \tilde{q}_j(\tilde{p}_j f) \right)
\end{aligned}$$

Hence $f = \lambda \left(\sum_i q_i(p_i f) + (-1)^{|f|} \sum_j \tilde{q}_j(\tilde{p}_j f) \right)$, and so λ is an isomorphism. This proves 2b).

We define super ring homomorphisms

$$\sigma : R \rightarrow \underline{End}_S(P)^o \quad \text{and} \quad \tau : R \rightarrow \underline{End}(Q_S)$$

by $p\sigma(r) := (-1)^{|p||r|}pr$ and $\tau(r)q := rq$.

The proof that $\sigma(r) \in \underline{End}({}_S P)$ is just like the proof for λ ; the proof that $\tau \in \underline{End}(Q_S)$ is trivial.

Both σ and τ preserve parity:

$$\begin{aligned} |p\sigma(r)| &= |(-1)^{|p||r|}pr| \\ &= |p| + |r| \\ |\tau(r)q| &= |rq| \\ &= |q| + |r| \end{aligned}$$

The proof that τ is a super ring homomorphism is a triviality. For the case of σ , we only note that since we are allowing $\underline{End}({}_S P)$ to act on the right instead of the left, we really have a super ring homomorphism from R to $\underline{End}({}_S P)^\circ$, as the reader will easily verify.

Hence σ and τ are super ring morphisms. The proof that σ, τ are isomorphisms is similar to the proof that λ is an isomorphism and is left to the reader. \square

The following proposition is complementary to the one just proved; it applies to finitely generated projective modules.

Proposition 3.1.4. *Let $P \in \mathfrak{M}_R$, $(R, P, Q, S; \alpha, \beta)$ the super Morita context associated to P . Then:*

- (1) *P is a finitely generated projective module iff β is onto.*
- (2) *Suppose P is a finitely generated projective module. Then:*

- a) *$\beta : P \otimes_R Q \rightarrow S$ is an (S, S) isomorphism.*
- b) *$Q \cong \underline{Hom}_R(P_R, R_R)$ as (R, S) -bimodules.*
- c) *$P \cong \underline{Hom}_R({}_R Q, {}_R R)$ as (S, R) -bimodules.*
- d) *$S \cong \underline{End}(P_R) \cong \underline{End}({}_R Q)^\circ$ as super rings.*

Proof. 1) β is onto iff there is an equation $1_S = \sum_l p_l'' q_l'' + \sum_m \tilde{p}_m'' \tilde{q}_m''$.

The super Dual Basis Lemma (Prop. 2.1.3) states that this is completely equivalent to P being finitely generated projective. The proof of 2) is completely analogous to the proof of the previous proposition, using the equation $1_S = \sum_l p_l'' q_l'' + \sum_m \tilde{p}_m'' \tilde{q}_m''$.

The only things that need to be noted are the following: the homomorphism $\lambda' : P \rightarrow \underline{Hom}_R({}_R Q, {}_R R)$ is given by $q\lambda'(p) := (-1)^{|q||p|}qp$, which is clearly parity preserving.

The map $Q \rightarrow \underline{Hom}_R(P_R, R_R)$ is just the identity, hence a parity preserving homomorphism. We also need to define ring homomorphisms:

$$\sigma' : S \rightarrow \underline{End}(P_R) \quad \text{and} \quad \tau' : S \rightarrow \underline{End}({}_R Q)^o$$

which are parity-preserving. This is obvious for $\sigma' := id$. We define τ' by $q \cdot \tau'(s) := (-1)^{|q||s|}qs$, which is clearly parity-preserving. The proof that τ' is an isomorphism proceeds as before. \square

3.2. The super Morita theorems. Finally, we come to the main results of this note. The first (“super Morita I”) states that tensoring with an R -progenerator P defines an equivalence of categories between \mathfrak{M}_R (resp. ${}_R\mathfrak{M}$) and \mathfrak{M}_S (resp. ${}_S\mathfrak{M}$), where $S = \underline{End}_R(P)$.

Theorem 3.2.1. *Let R be a super ring, P_R a progenerator, and $(R, P, Q, S; \alpha, \beta)$ the super Morita context associated with P_R . Then:*

- (1) $-\otimes_R Q : \mathfrak{M}_R \rightarrow \mathfrak{M}_S$ and $-\otimes_S P : \mathfrak{M}_S \rightarrow \mathfrak{M}_R$ are mutually inverse category equivalences.
- (2) $P \otimes_R - : {}_R\mathfrak{M} \rightarrow {}_S\mathfrak{M}$ and $Q \otimes_S - : {}_S\mathfrak{M} \rightarrow {}_R\mathfrak{M}$ are mutually inverse category equivalences.

Proof. Let M be a right R -module. Then

$$\begin{aligned} (M \otimes_R Q) \otimes_S P &\cong M \otimes_R (Q \otimes_S P) \\ &\cong M \otimes_R R \\ &\cong M \end{aligned}$$

where the first isomorphism is the canonical associativity isomorphism of the super tensor product induced by $(m \otimes q) \otimes p \mapsto m \otimes (q \otimes p)$, the second isomorphism follows from Prop. 3.1.3, and the third is the canonical isomorphism induced by $m \otimes r \mapsto mr$. All of these isomorphisms are clearly functorial in M , whence it follows that the composition of $-\otimes_S P$ with $-\otimes_R Q$ is naturally equivalent to the identity.

If N is a right S -module, the same proof goes through, switching the roles of R and S , as well as those of P and Q , and instead of Prop. 3.1.3, we invoke Prop. 3.1.4 to conclude that $N \otimes_S (P \otimes_R Q) \cong N \otimes_S S$.

Again, all isomorphisms are functorial in M , whence it follows that the composition of $-\otimes_R Q$ with $-\otimes_S P$ is naturally equivalent to the identity. Hence we have proven that $P \otimes_R -$ and $Q \otimes_S -$ are mutually inverse category equivalences.

The proof of part 2) is completely analogous and is left to the reader. \square

We now make a useful definition.

Definition 3.2.2. Let A, B be super rings. An (A, B) -bimodule C is *faithfully balanced* if the natural maps $A \rightarrow \underline{\text{End}}(C_B)$ and $B \rightarrow \underline{\text{End}}({}_A C)^\circ$ are both isomorphisms of super rings.

By Props. 3.1.3 and 3.1.4, if P_R is a progenerator, ${}_S P_R$ and ${}_R Q_S$ are faithfully balanced bimodules.

The following proposition shows that the roles of P and Q in the Morita theory are completely symmetric.

Proposition 3.2.3. *Suppose P_R is a progenerator. Then ${}_S P, {}_R Q, Q_S$ are also progenerators, and α, β are isomorphisms.*

Proof. First, we prove that Q_S is a progenerator. By Prop. 3.1.3(2), we have $\underline{\text{Hom}}_S(Q_S, S_S) \cong P$ and $\underline{\text{End}}(Q_S) \cong R$. As α, β are surjective, we see from Prop. 3.1.3(1) and 3.1.4(1), applied to Q_S , that Q_S is a progenerator.

The proof for ${}_R Q$ is analogous to that for ${}_S P$ and so we only do the case of ${}_S P$. We denote $\underline{\text{End}}_S({}_S P)$ by S' . P is an $(S, S')^\circ$ -bimodule. By 2d of Prop. 3.1.3 we have a super ring isomorphism $\sigma : R \cong (S')^\circ$. Recalling that σ is defined by the right action of R , the natural equivalence of categories $\sigma^{-1} : \mathfrak{M}_{(S')^\circ} \rightarrow \mathfrak{M}_R$ induced by the isomorphism σ^{-1} clearly sends $P_{(S')^\circ}$ to P_R . By hypothesis P_R is an R -progenerator, hence $P_{(S')^\circ}$ is an $(S')^\circ$ -progenerator. The S -dual $P^\vee := \underline{\text{Hom}}_S({}_S P, {}_S S)$ is then an $((S')^\circ, S)$ -bimodule in the usual way, and again the equivalence of categories $\sigma^{-1} : (S')^\circ \mathfrak{M} \rightarrow {}_R \mathfrak{M}$ sends $(S')^\circ P^\vee$ to ${}_R P^\vee$.

We want to show the pairings $\alpha' : P \otimes_{(S')^\circ} P^\vee \rightarrow S$ and $\beta' : P^\vee \otimes_S P \rightarrow (S')^\circ$ are epimorphisms. Identifying $\mathfrak{M}_{(S')^\circ}$ with \mathfrak{M}_R and $(S')^\circ \mathfrak{M}$ with ${}_R \mathfrak{M}$, and identifying the (R, S) -modules Q and P^\vee via 2c of Prop. 3.1.3 we see that this is equivalent to $\beta : P \otimes_R Q \rightarrow S$ and $\alpha : Q \otimes_S P \rightarrow R$ being epimorphisms. Since P_R is a progenerator, that is indeed the case.

Now we apply the functor \cdot° to convert everything to right modules. ${}_S P_{(S')^\circ}$ becomes ${}_S P_{S^\circ}$, and since $P^\dagger := \underline{\text{Hom}}_{S^\circ}(P_{S^\circ}, S_{S^\circ}^\circ) = P^\vee$, $(S')^\circ P_S^\vee$ becomes ${}_S P_{(S')^\circ}^\dagger$. The (S, S) (resp. $((S')^\circ, (S')^\circ)$) epimorphisms $\alpha' : P \otimes_{(S')^\circ} P^\vee \rightarrow S$ and $\beta' : P^\vee \otimes_S P \rightarrow (S')^\circ$ become (S°, S°) (resp. (S', S')) epimorphisms $\alpha' : P \otimes_{(S')^\circ} P^\vee \rightarrow S$ and $\beta' : P^\vee \otimes_S P \rightarrow (S')^\circ$. By 3.1.3 and 3.1.4, P_{S° is a progenerator. Since being a progenerator is a categorical property (Prop. 2.3.4) we finally conclude that ${}_S P$ is a progenerator as well. \square

Hence if one of $P_R, {}_R Q, {}_S P, Q_S$ are R -progenerators (resp. S -progenerators), the rest of them are too.

Now we prove the second of the main Morita theorems (“super Morita II”), a converse to the first. It states that *every* Morita equivalence between two categories of super modules is (up to natural equivalence) of the form given in the first Morita theorem: i.e., by tensoring with a progenerator.

Theorem 3.2.4. *Let R, S be two super rings, and*

$$F : \mathfrak{M}_R \rightarrow \mathfrak{M}_S$$

$$G : \mathfrak{M}_S \rightarrow \mathfrak{M}_R$$

be mutually inverse category equivalences. Let $Q = F(R_R)$, $P = G(S_S)$. Then there are natural bimodule structures $P = {}_S P_R, Q = {}_R Q_S$, which yield functor isomorphisms $F \cong - \otimes_R Q$ and $G \cong - \otimes_S P$.

Proof. Since $G(S) = P$, $\underline{\text{End}}(S_S) \cong \underline{\text{End}}(P_R)$ as super rings, via $f \mapsto G(f)$. It is easily seen that $\underline{\text{End}}(S_S) \cong S$ as super rings via the morphism $f \mapsto f(1)$ (the inverse being given by $s \mapsto g_s$, where $g_s(x) = sx$). Hence we have a natural (S, R) -bimodule structure ${}_S P_R$. Via a similar argument for Q , we have a natural (R, S) -bimodule structure ${}_R Q_S$. As S_S is a progenerator in \mathfrak{M}_S , P_R is a progenerator in \mathfrak{M}_R by Prop. 2.3.4.

$\underline{\text{Hom}}_R(P, R)$ has its usual left R -module structure induced by left multiplication in R , and a right S -module structure defined by QSP -associativity. We verify that $\underline{\text{Hom}}_R(P, R) \cong Q$ as right S -modules:

$$\underline{\text{Hom}}_R(P, R) \cong \underline{\text{Hom}}_S(F(P), F(R)) \cong \underline{\text{Hom}}_S(S_S, Q_S) \cong Q$$

Hence, the super Morita context associated to P_R is $(R, P, Q, S; \alpha, \beta)$, where α, β are the pairings defined previously. Now the first Morita theorem applies; it remains to show that F is naturally equivalent to the functor $- \otimes_R Q$.

Given any $M \in \mathfrak{M}_R$,

$$F(M) \cong \underline{\text{Hom}}_S(S_S, F(M)) \cong \underline{\text{Hom}}_R(P_R, M_R)$$

whence $F \cong \underline{\text{Hom}}_R(P_R, -) \cong - \otimes_R Q$ by Lem. 4.1.5. Similarly, we have $G \cong \underline{\text{Hom}}_S(Q_S, -) \cong - \otimes_S P$. \square

We have one more theorem (“super Morita III”) which characterizes the isomorphism classes of equivalences between super module categories. In order to state this theorem, we require the following:

Definition 3.2.5. Let R, S be super rings. An (S, R) -bimodule ${}_S P_R$ is an (S, R) -progenerator if ${}_S P_R$ is faithfully balanced and P_R is an R -progenerator.

We may now state the third of our Morita theorems:

Theorem 3.2.6. *Let R and S be two super rings. Then the isomorphism classes of category equivalences $\mathfrak{M}_S \rightarrow \mathfrak{M}_R$ are in one-to-one correspondence with the isomorphism classes of (S, R) -progenerators. Composition of category equivalences corresponds to tensor products of these progenerators.*

Proof. Each (S, R) -progenerator yields a category equivalence $- \otimes_S P : \mathfrak{M}_S \rightarrow \mathfrak{M}_R$. The isomorphism class of this equivalence depends only on the isomorphism class of the (S, R) -bimodule P . Conversely, suppose $G : \mathfrak{M}_S \rightarrow \mathfrak{M}_R$ is a category equivalence. Then $P := G(S_S)$ is an (S, R) -progenerator, as in the proof of the second Morita theorem. The isomorphism class of the (S, R) -bimodule P clearly depends only on the isomorphism class of the equivalence G , proving the first statement. If ${}_R P'_T$ is an (R, T) -progenerator, the composition of equivalences $\mathfrak{M}_S \rightarrow \mathfrak{M}_R \rightarrow \mathfrak{M}_T$ is given by $- \otimes_S (P \otimes_R P')$, whence the second conclusion. \square

Remark: One can state versions of Morita II and Morita III for categories of left modules, and prove them in exactly the same way as we have done for categories of right modules, or one can use \cdot^o to reduce to the right-module versions of the theorems. This is left to the reader.

4. APPLICATION: SUPER AZUMAYA ALGEBRAS

4.1. Super Azumaya algebras. Let k be a field. Recall that a ungraded k -algebra B is said to be *central* if its center is k , and *simple* if B has no non-trivial two-sided ideals. Those algebras which are finite-dimensional and central simple over k are characterized by the Artin-Wedderburn theorem:

Theorem 4.1.1. *Let A be an algebra over a field k which is finite dimensional as a k -vector space. Then A is central simple over k if and only if the map:*

$$\begin{aligned} A \otimes A^o &\rightarrow \text{End}_k(A) \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism of k -algebras.

This theorem generalizes to the context of superalgebras. For this we must recall some basic definitions.

The supercenter of a super ring A is the sub-super ring:

$$Z(A) := \{x \in A : ax = (-1)^{|a||x|}xa \text{ for all homogeneous } a \in A\}$$

From now on, we suppose that R is a supercommutative ring, i.e. $rr' = (-1)^{|r||r'|}r'r$ for all homogeneous $r, r' \in R$. An R -superalgebra is a super ring A with a super ring morphism $i : R \rightarrow A$ such that $i(R) \subseteq Z(A)$.

The opposite A^o is also an R -superalgebra in a natural way; since $i(R) \subseteq Z(A)$, and R is supercommutative, $i^o : R \rightarrow A^o$ is a super ring morphism (here i^o denotes the map $R \rightarrow A^o$ that agrees with i as a map of sets).

If A, B are R -superalgebras, the tensor product $A \otimes_R B$ possesses a natural structure of R -superalgebra, the multiplication being given by:

$$(a \otimes b) \cdot (c \otimes d) := (-1)^{|b||c|}ac \otimes bd$$

Given any R -superalgebra A , there is a natural R -superalgebra morphism $\phi : A \otimes_R A^o \rightarrow \underline{\text{End}}_R(A)$ given by:

$$a \otimes b \mapsto (x \mapsto (-1)^{|b||x|}axb)$$

From now on, we will denote the superalgebra $A \otimes_R A^o$ by A^e for the sake of brevity.

We say that a k -superalgebra A is *central* if its supercenter equals k , and A is *simple* if A has no non-trivial two-sided homogeneous ideals. The super Artin-Wedderburn theorem (see, e.g. [10]) then states that:

Theorem 4.1.2. *Let A be a superalgebra over a field k , $\text{char}(k) \neq 2$, which is finite dimensional as a k -super vector space. Then A is central*

simple over k if and only if $\phi : A^e \rightarrow \underline{\text{End}}_k(A)$ is an isomorphism of k -superalgebras.

In ordinary commutative algebra, the notion of central simple algebra over a field k has been generalized to the category of algebras over a commutative ring by adopting the conclusion of the Artin-Wedderburn theorem as a definition. The resulting objects are called *Azumaya algebras* (one may see [6] for more on the basics of ungraded Azumaya algebras).

We define the corresponding super notion as follows.

Definition 4.1.3. Let A/R be a superalgebra over a supercommutative ring R . We say that A is a *super Azumaya algebra* over R iff A is a faithful, finitely generated projective R -module, and the natural morphism $\phi : A^e \rightarrow \underline{\text{End}}_R(A)$ is an isomorphism of R -superalgebras.

Example. Let k be an algebraically closed field of characteristic $\neq 2$. We define the super skew field ([4]) to be:

$$\mathbb{D} := k[\theta], \theta \text{ odd}, \theta^2 = -1$$

In [4] it is shown that \mathbb{D} is central simple over k , hence a super Azumaya algebra over k , and that (the Brauer equivalence class of) \mathbb{D} generates the super Brauer group of k . For the definition of the super Brauer group of a field and basic results, see again [4]; the original source for this material is [11].

By definition a super Azumaya algebra A is a progenerator, hence, by the results of the previous section, there is a Morita equivalence between ${}_R\mathfrak{M}$ and ${}_{A^e}\mathfrak{M}$). Our aim in this section is to make this Morita equivalence even more explicit by expressing it in terms of the notion of *supercommutant*.

We note that the concepts of (A, A) -bimodule and left A^e -module are completely equivalent: if M is a left A^e module, we define an (A, A) -bimodule structure by:

$$\begin{aligned} a \cdot m &:= (a \otimes 1) \cdot m \\ m \cdot a &:= (-1)^{|a||x|} (1 \otimes a) \cdot m \end{aligned}$$

where $a \in A$, $m \in M$ are homogeneous.

Conversely, if N is an (A, A) -bimodule, we define a left A^e -module structure on N via:

$$(a \otimes b) \cdot n := (-1)^{|b||n|} a \cdot n \cdot b$$

It's readily seen that these correspondences are compatible with (A, A) -morphisms and A^e -morphisms respectively. Hence, there is a natural equivalence of categories between the category of (A, A) -bimodules and that of left A^e -modules.

We begin with a super version of a standard fact from the theory of modules (cf. Exercise 20 of [8], Ch. 2):

Lemma 4.1.4. *Let R be a super ring (not necessarily commutative), and $P, B \in \mathfrak{M}_R$. Define the morphism of superabelian groups $\sigma_{P,B} : P^* \otimes_R B \rightarrow \underline{\text{Hom}}_R(P, B)$ by:*

$$[\sigma_{P,B}(f \otimes b)](x) := (-1)^{|x||b|} f(x) \cdot b$$

Then if P is a finitely generated projective R -module, $\sigma_{P,B}$ is an isomorphism. Furthermore, $\sigma_{P,-}$ is functorial in B : given a R -morphism $k : B \rightarrow B'$, the diagram of superabelian groups:

$$\begin{array}{ccc} P^* \otimes_R B & \xrightarrow{\sigma_{P,B}} & \underline{\text{Hom}}_R(P, B) \\ \downarrow \text{id} \otimes k & & \downarrow k_* \\ P^* \otimes_R B' & \xrightarrow{\sigma_{P,B'}} & \underline{\text{Hom}}_R(P, B') \end{array}$$

is commutative.

Proof. First, one readily checks that $\sigma_{P,B}$ is a parity-preserving homomorphism of abelian groups, so that it is indeed an SAb -morphism. First we consider the case where P is free of finite rank, beginning with the case where P is free of rank $1|0$ or $0|1$. So suppose that $P = R$ as R -modules. Then one may check readily (as in the ungraded case) that $\underline{\text{Hom}}_R(R, B) \cong B$ in SAb , via $\phi \mapsto \phi(1)$. By composing this isomorphism with $\sigma_{R^*,B}$, we have an SAb -morphism $\sigma'_{R^*,B} : R^* \otimes_R B \rightarrow B$, with $\sigma'_{R^*,B}(f \otimes b) := f(1)b$.

We are now reduced to proving that $\sigma'_{R^*,B}$ is an isomorphism; this follows after one checks that the inverse homomorphism $\sigma'^{-1}_{R^*,B}$ is given by $b \mapsto 1^* \otimes b$, where 1^* is the functional in R^* dual to $1 \in R$.

The case of $P = \Pi R$ is completely analogous, but one must keep careful track of the parity reversals.

We have an *SAb* isomorphism $\underline{Hom}(\Pi R, B) \cong \Pi B$ via $\phi \mapsto \Pi[\phi \Pi(\Pi^2 1)] = \Pi[\phi(\Pi(1))]$. This is the composition of three isomorphisms: the odd isomorphism $\underline{Hom}_R(\Pi R, B) \rightarrow \underline{Hom}_R(\Pi^2 R, B) = \underline{Hom}_R(R, B)$ given by $\phi \mapsto \phi \Pi$ (see Lem 2.3.1), the above (even) isomorphism $Hom_R(R, B) \cong B$, and finally the odd isomorphism $\Pi(id) : B \rightarrow \Pi B$.

Composing this with $\sigma_{(\Pi R)^*, B}$, we have a new morphism $\sigma'_{(\Pi R)^*, B} : (\Pi R)^* \otimes B \rightarrow \Pi B$ with $\sigma'_{(\Pi R)^*, B}(f \otimes b) = (-1)^{|b|} \Pi[f(\Pi 1)b]$, and it is again enough to show that $\sigma'_{(\Pi R)^*, B}$ is an isomorphism. One checks that the inverse homomorphism $\sigma'^{-1}_{(\Pi R)^*, B}$ is given by $\Pi b \mapsto (\Pi(1))^* \otimes b$.

Now let P be a finite-rank free module: $P = \bigoplus_j P_j$, where there are only finitely many j , and each P_j is either isomorphic to R or ΠR . We use the standard fact that \underline{Hom} is compatible with direct sums:

$$\underline{Hom}(\bigoplus_j P_j, M) \cong \prod_j \underline{Hom}(P_j, M) \cong \bigoplus_j \underline{Hom}(P_j, M)$$

The first isomorphism is the universal property of the direct sum, hence is natural. The second isomorphism is the natural identification of a finite direct product with a finite direct sum (here we use the hypothesis that F has finite rank).

One may check that $\sigma_{-, B}$ is compatible with these identifications, in the sense that:

$$\begin{array}{ccc} P_j^* \otimes_R B & \xrightarrow{\sigma_{P_j, B}|_{P_j}} & \underline{Hom}_R(P_j, B) \\ i_j^* \otimes id \uparrow & & \uparrow i_j^* \\ P^* \otimes_R B & \xrightarrow{\sigma_{P, B}} & \underline{Hom}_R(P, B) \end{array}$$

where i_j is the inclusion of the j th summand into the direct sum. Hence, taking the direct sum over all (finitely many) indices j , we have:

$$\begin{array}{ccc} \bigoplus_j (P_j^* \otimes_R B) & \xrightarrow{\bigoplus_j \sigma_{P_j, B}|_{P_j}} & \bigoplus_j \underline{Hom}_R(P_j, B) \\ \bigoplus_j i_j^* \otimes id \uparrow & & \uparrow \bigoplus_j i_j^* \\ P^* \otimes_R B & \xrightarrow{\sigma_{P, B}} & \underline{Hom}_R(P, B) \end{array}$$

For any j , we have either $P_j \cong R$ or $P_j \cong \Pi R$. By what we proved earlier, $\sigma_{P,B}|_{P_j}$ is an isomorphism for each j , thus their direct sum $\bigoplus_j \sigma_{P,B}|_{P_j}$ is also an isomorphism. The morphisms $\bigoplus_j (i_j^* \otimes id)$ and $\bigoplus_j i_j^*$ are just the identity maps, hence isomorphisms. By commutativity of the diagram, it follows that $\sigma_{P,B}$ is an isomorphism as well.

In turn, we reduce the general case to the case of a finite-rank free module as follows. Since P is finitely generated projective, there exists a free module F of finite rank and a split epimorphism $\pi : F \rightarrow P$ with splitting $i : P \rightarrow F$. As $i \bullet \pi = id_P$, so $i^* \circ \pi^* = id_{P^*}$. Hence π^* is injective, i^* surjective.

Let $\underline{\pi}^*$ denote the morphism from $\underline{Hom}_R(P, B)$ to $\underline{Hom}_R(F, B)$ induced by π , and \underline{i}^* the morphism $\underline{Hom}_R(F, B)$ to $\underline{Hom}_R(P, B)$ induced by i . Then the same considerations as before give $\underline{i}^* \circ \underline{\pi}^* = id_{\underline{Hom}_R(P, B)}$. Consequently, $\underline{\pi}^*$ is injective, \underline{i}^* surjective.

We claim the following diagrams are commutative:

$$\begin{array}{ccc} F^* \otimes_R B & \xrightarrow{\sigma_{F,B}} & \underline{Hom}_R(F, B) \\ \uparrow \pi^* \otimes id_B & & \uparrow \underline{\pi}^* \\ P^* \otimes_R B & \xrightarrow{\sigma_{P,B}} & \underline{Hom}_R(P, B) \end{array}$$

$$\begin{array}{ccc} F^* \otimes_R B & \xrightarrow{\sigma_{F,B}} & \underline{Hom}_R(F, B) \\ \downarrow i^* \otimes id_B & & \downarrow \underline{i}^* \\ P^* \otimes_R B & \xrightarrow{\sigma_{P,B}} & \underline{Hom}_R(P, B) \end{array}$$

We will check commutativity of the first. Let $y \in F, f \otimes b \in P^* \otimes B$. Then:

$$\begin{aligned} [\underline{\pi}^* \circ \sigma_{P,B}(f \otimes b)](y) &= [\sigma_{P,B}(f \otimes b)](\pi(y)) \\ &= (-1)^{|\pi(y)||b|} f(\pi(y)) \cdot b \\ &= (-1)^{|y||b|} \pi^*(f)(y) \cdot b \\ &= [\sigma_{F,B}(\pi^*(f) \otimes b)](y) \\ &= [(\sigma_{F,B} \circ (\pi^* \otimes id))(f \otimes b)](y) \end{aligned}$$

which is the statement that the first diagram commutes.

Commutativity of the second diagram is analogous and is left to the reader.

The lemma follows easily from this, combined with the fact that $\sigma_{F,B}$ is an isomorphism (F is a free module). For injectivity of $\sigma_{P,B}$: suppose $a \in P^* \otimes B$ and $\sigma_{P,B}(a) = 0$. But commutativity of the first diagram gives $(\pi^* \otimes id_B)(a) = \sigma_{F,B}^{-1} \circ \underline{\pi}^* \circ \sigma_{P,B}(a)$ hence $(\pi^* \otimes id_B)(a) = 0$. Since $\pi^* \otimes id_B$ is injective, $a = 0$.

For surjectivity of $\sigma_{P,B}$: suppose $c \in \underline{Hom}_R(P, B)$. Then since \underline{i}^* is onto, $c = \underline{i}^*(c')$ for some $c' \in \underline{Hom}_R(F, B)$. Let $a = i^* \circ \sigma_{F,B}^{-1}(c')$. Then by commutativity of the second diagram, $\sigma_{P,B}(a) = c$.

It remains to prove functoriality in B . Let $k : B \rightarrow B'$ be a morphism, and let $x \in P$.

$$\begin{aligned} [k_* \circ \sigma_{P,B}(f \otimes b)](x) &= k[\sigma_{P,B}(f \otimes b)(x)] \\ &= k((-1)^{|b||x|} f(x) \cdot b) \\ &= (-1)^{|k(b)||x|} f(x) \cdot k(b) \\ &= [\sigma_{P,B'}(f \otimes k(b))](x) \\ &= [(\sigma_{P,B'} \circ (id \otimes k))(f \otimes b)](x) \end{aligned}$$

which is the statement that the given diagram commutes. \square

Remark. We will need the following additional property of $\sigma_{P,B}$ for our applications. If P is also a right S -module for some super ring S , $\underline{Hom}_R(P, B)$ has a natural structure of left S -module via the “pullback” action:

$$(sf)(x) := (-1)^{|f(x)||s|} f(xs)$$

P^* also has a left S -module structure by the same formula, which induces a left S -module structure on $P^* \otimes_R B$. It is easily seen that with these left S -module structures, $\sigma_{P,B}$ is a morphism (and by the lemma, an isomorphism) in ${}_S\mathfrak{M}$.

Of course, we have an analogous result for right R -modules. Since we need this for the proof of “super Morita II”, we formulate it explicitly:

Lemma 4.1.5. *Let R be a super ring (not necessarily commutative), and $P, B \in {}_R\mathfrak{M}$. Define the morphism of superabelian groups $\sigma_{B,P} : B \otimes_R P^* \rightarrow \underline{Hom}_R(P, B)$ by:*

$$[\sigma_{B,P}(b \otimes f)](x) := b \cdot f(x)$$

Then if P is a finitely generated projective R -module, $\sigma_{B,P}$ is an isomorphism. Furthermore, $\sigma_{-,P}$ is functorial in B : given a R -morphism $k : B \rightarrow B'$, the diagram of superabelian groups:

$$\begin{array}{ccc} P^* \otimes_R B & \xrightarrow{\sigma_{B,P}} & \underline{\text{Hom}}_R(P, B) \\ \downarrow \text{id} \otimes k & & \downarrow k_* \\ P^* \otimes_R B' & \xrightarrow{\sigma_{B',P}} & \underline{\text{Hom}}_R(P, B') \end{array}$$

is commutative.

To prove this, one can, as usual, either adapt the proof of Lem. 4.1.4 to the category of right modules, or use the functor \cdot^o to reduce everything to the case of left modules.

4.2. The supercommutant.

Definition 4.2.1. Let M be an (A, A) -bimodule. The *supercommutant* of M is the (R, R) -bimodule M^A generated by:

$$\{m \in M : am = (-1)^{|a||m|}ma \ \forall \text{ homogeneous } a \in A\}.$$

Equivalently, interpreting M as a left A^e -module, we see that M^A may be defined in terms of the A^e -action, as the (R, R) -bimodule generated by:

$$\{m \in M : (a \otimes 1)m = (1 \otimes a)m \ \forall \text{ homogeneous } a \in A\}$$

M^A so defined is indeed an (R, R) -bimodule: as $i(R) \subseteq Z(A)$, the action of R on M commutes with the action of A . Let $a \in A$, $r \in R$, and suppose $m \in M^A$. Then:

$$\begin{aligned}
a(rm) &= (ar)m \\
&= (-1)^{|a||r|}(ra)m \\
&= (-1)^{|a||r|+|ra||m|}m(ra) \\
&= (-1)^{|a||r|+|ra||m|}(mr)a \\
&= (-1)^{|ra||m|}(rm)a
\end{aligned}$$

which is the statement that $rm \in M^A$. That $mr \in M^A$ may be checked in completely analogous fashion.

We have the following functoriality property: if $f : M \rightarrow N$ is an A^e -morphism, then clearly $f(M^A) \subseteq N^A$ and $f' := f|_{M^A} : M^A \rightarrow N^A$ is an (R, R) -morphism. Clearly $(f \circ g)' = f' \circ g'$ and $(id_M)' = id_{M^A}$.

It follows that the operation of taking the supercommutant may be regarded as a functor $-^A : {}_{A^e}\mathfrak{M} \rightarrow {}_R\mathfrak{M}$, where $M \mapsto M^A, f \mapsto f'$ for any A^e -module M and A^e -morphism $f : M \rightarrow N$.

We now show

Theorem 4.2.2. *$\underline{Hom}_{A^e}(A, M)$ is naturally isomorphic to M^A as a left R -module.*

Proof. We define the isomorphism $F : \underline{Hom}_{A^e}(A, M) \rightarrow M^A$ as follows. Suppose $f : A \rightarrow M$ is an A^e -homomorphism. Then:

$$\begin{aligned}
f(a) &= f((a \otimes 1) \cdot 1) \\
&= a \cdot f(1) \cdot 1 \\
&= a \cdot f(1)
\end{aligned}$$

$$\begin{aligned}
f(a) &= f((1 \otimes a) \cdot 1) \\
&= (-1)^{|f(1)||a|} \cdot f(1) \cdot a \\
&= (-1)^{|f(1)||a|} f(1) \cdot a
\end{aligned}$$

hence $f(1) \in M^A$. Thus we have a parity-preserving map:

$$\begin{aligned}
F : \underline{Hom}_{A^e}(A, M) &\rightarrow M^A \\
f &\mapsto f(1)
\end{aligned}$$

Recalling that the R -module structure on $\underline{Hom}_{A^e}(A, M)$ is defined by the “pullback” action:

$$(rf)(x) := (-1)^{|r||f(x)|} f(xr)$$

it is readily checked that F is an R -morphism. Conversely, suppose $m \in M^A$. Then we define a map $g_m : A \rightarrow M$ by:

$$g_m(x) = x \cdot m$$

We check that g_m so defined is indeed an A^e -homomorphism:

$$\begin{aligned} g_m(a \otimes b \cdot x) &= g((-1)^{|b||x|} axb) \\ &= (-1)^{|b||x|} axbm \\ &= (-1)^{|b|(|x|+|m|)} axmb \quad (\text{since } m \in M^{A^e}) \\ &= (-1)^{|b||g_m(x)|} a \cdot g_m(x) \cdot b \\ &= (a \otimes b) \cdot g_m(x) \end{aligned}$$

Hence $m \mapsto g_m$ is a parity-preserving map $G : M^A \rightarrow \underline{Hom}_{A^e}(A, M)$, and it's easily seen that G is inverse to F . Since F is an R -morphism, so is G .

We now verify the naturality statement of the theorem: that $F : \underline{Hom}_{A^e}(A, M) \rightarrow M^A$ and $G : M^A \rightarrow \underline{Hom}_{A^e}(A, M)$ are functorial in M .

Let $h : M \rightarrow N$ be an A^e -morphism. Then h induces the R -morphism $h_* : \underline{Hom}_{A^e}(A, M) \rightarrow \underline{Hom}_{A^e}(A, N)$ by $h_*(f) = h \circ f$.

The statement that F is a natural transformation is the equality $F \circ h_* = h' \circ F$, where h' denotes the restriction of h to M^A .

$$\begin{aligned} F \circ h_*(f) &= F \circ (h \circ f) \\ &= (h \circ f)(1) \\ &= h(f(1)) \\ &= h' \circ F(f) \end{aligned}$$

The proof that G is a natural transformation is completely analogous: we verify the equality $G \circ h' = h_* \circ G$.

$$\begin{aligned}
G \circ h'(m) &= G(h(m)) \\
&= g_{h(m)} \\
&= h \circ g_m \\
&= h_* \circ G(m)
\end{aligned}$$

□

4.3. The main result. Putting all this together, we have our main result:

Theorem 4.3.1. *Let R be a supercommutative ring, A/R a super Azumaya algebra. Then the functors:*

$$\begin{aligned}
A \otimes_R - : {}_R\mathfrak{M} &\rightarrow {}_{A^e}\mathfrak{M} \\
-A : {}_{A^e}\mathfrak{M} &\rightarrow {}_R\mathfrak{M}
\end{aligned}$$

are mutually inverse category equivalences.

Proof. By “super Morita I” (Thm. 3.2.1), we have that $A \otimes_R - :$ and $A^* \otimes_{A^e} -$ are mutually inverse category equivalences (after composing with the obvious category equivalence ${}_S\mathfrak{M} \rightarrow {}_{A^e}\mathfrak{M}$ induced by the isomorphism $\phi : A^e \rightarrow \underline{\text{End}}_R(A) = S$). Let A^\vee denote $\underline{\text{Hom}}_{A^e}(A, A^e)$. We claim there is a sequence of natural isomorphisms:

$$\begin{aligned}
A^* \otimes_{A^e} M &\cong A^\vee \otimes_{A^e} M \\
&\cong \underline{\text{Hom}}_{A^e}(A, M) \\
&\cong M^A
\end{aligned}$$

By part 2b) of Lem. 3.1.4 $A^* \cong A^\vee$ as an (R, A^e) -bimodule; hence the first isomorphism exists and is obviously functorial in M . Note that the R -action on A^\vee is given by $x(rf) := (-1)^{|r||f(x)|}(xr)f$. By Lemma 3.2.3, A^\vee is a projective A^e -module. Hence, by Lemma 4.1.4 and the following Remark, the second isomorphism (of left R -modules) exists and is functorial in M . By Thm. 4.2.2, the third isomorphism (of left R -modules) exists and is also functorial in M . We conclude that the

identification of $A^* \otimes_{A^e} M$ with M^A is functorial in M , hence we have shown that the functor $-^A$ is naturally equivalent to $A^* \otimes_{A^e} -$, so is also a functor inverse to $A \otimes_R -$, as desired.

□

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